A few properties of Ramanujan cubic polynomials and Ramanujan cubic polynomials of the second kind

Beata BAJORSKA-HARAPIŃSKA,
Mariusz PLESZCZYŃSKI, Damian SŁOTA
and Roman WITUŁA

Abstract. In this paper we collect most of the known and add some new algebraic and analytic properties of Ramanujan cubic polynomials (RCP) and Ramanujan cubic polynomials of the second kind (RCP2). One of our goals was to systematize the current knowledge on RCPs fulfilling the list of properties with some new ones, involving the new representation of RCPs. Our next goal was to examine differences, similarities and connections between RCPs and RCP2s. The last goal was to discuss some asymptotic properties of RCP2.

Keywords: Ramanujan cubic polynomial, Shevelev parameter, Shanks cubic polynomial.

2010 Mathematics Subject Classification: 11C08, 11B83, 33B10.

1. Introduction

In this paper we collect (most of) the known and add some new algebraic and analytic properties of two interesting families of cubic polynomials – namely Ramanujan cubic polynomials (RCP) (see Definition 1.1 below) and the associated family of so-called Ramanujan cubic polynomials of the second kind (RCP2) (see Definition 1.3 below). Shevelev in [8] have started the regular studies of RCPs, their roots and identities they imply. Since then these polynomials are studied in many directions and from different points of view. The family of RCP2s was introduced by Wituła in [11] as the supplement (and the opposite in the same time) of RCPs, but it turned out that these polynomials are interesting on their own.

B. Bajorska-Harapińska, M. Pleszczyński, D. Słota, R. Wituła
Institute of Mathematics, Silesian University of Technology, Kaszubska 23, 44-100 Gliwice, Poland,
e-mail: {beata.bajorska, mariusz.pleszczynski, damian.slota, roman.witula}@polsl.pl

R. Wituła, B. Bajorska-Harapińska, E. Hetmaniok, D. Słota, T. Trawiński (eds.), *Selected Problems on Experimental Mathematics*. Wydawnictwo Politechniki Śląskiej, Gliwice 2017, pp. 181–200.

One of our goals was to systematize the current knowledge on RCPs fulfilling the list of properties with some new ones, like the new representation of RCPs (see Theorem 1.2 and Theorem 4.1 point 1.) connected with so-called Shevelev's parameter (see formula (9)). On a base of it we have obtained the best possible characterization of RCPs with the same Shevelev's parameter (see Theorem 1.2 point 2.), which improves the result given in [10]. Moreover, it helped us to understand the base and nature of some identities discovered by Shevelev (see Theorem 4.1 point 3.).

Our next goal was to examine differences, similarities and connections between RCPs and RCP2s, which allow to extend the knowledge on properties of both families of polynomials. In particular, we gave an algorithm kept in a spirit of Ramanujan, namely how to obtain RCP2 with roots being cubic roots of roots of some RCP (see Theorem 3.1).

The last goal was to discuss some asymptotic properties of RCP2.

1.1. RCPs, their representations and forms of roots

In *Journal of the Indian Mathematical Society* Srinivasa Ramanujan asked the questions about the proof the following identities

Q682, JIMS VII, 1915
$$\left(\frac{1}{9}\right)^{\frac{1}{3}}-\left(\frac{2}{9}\right)^{\frac{1}{3}}+\left(\frac{4}{9}\right)^{\frac{1}{3}}=(2^{\frac{1}{3}}-1)^{\frac{1}{3}},$$

Q524, JIMS VI, 1914
$$\left(\cos\frac{2\pi}{7}\right)^{\frac{1}{3}}+\left(\cos\frac{4\pi}{7}\right)^{\frac{1}{3}}+\left(\cos\frac{8\pi}{7}\right)^{\frac{1}{3}}=\left(\frac{5-3\cdot 7^{\frac{1}{3}}}{2}\right)^{\frac{1}{3}},$$

Q524, JIMS VI, 1914
$$\left(\cos\frac{2\pi}{9}\right)^{\frac{1}{3}}+\left(\cos\frac{4\pi}{9}\right)^{\frac{1}{3}}+\left(\cos\frac{8\pi}{9}\right)^{\frac{1}{3}}=\left(\frac{3\cdot 9^{\frac{1}{3}}-6}{2}\right)^{\frac{1}{3}}.$$

More on above two and the other 56 Ramanujan's questions can be found e.g. in [5]. Studying the nature of the above equalities let Ramanujan prove a more general result (the proof appeared in the second of his famous *Notebooks*)

Theorem A (Ramanujan, [6]). *Let α, β, γ denote the roots of the cubic equation*

$$x^3 - ax^2 + bx - 1 = 0. \tag{1}$$

Then, for a suitable determination of roots

$$\alpha^{\frac{1}{3}}+\beta^{\frac{1}{3}}+\gamma^{\frac{1}{3}}=(a+6+3t)^{\frac{1}{3}}$$

and

$$(\alpha\beta)^{\frac{1}{3}}+(\beta\gamma)^{\frac{1}{3}}+(\gamma\alpha)^{\frac{1}{3}}=(b+6+3t)^{\frac{1}{3}},$$

where

$$t^3-3(a+b+3)t-(ab+6(a+b)+9)=0.$$

For the proof see e.g. [3]. Regular studies on a generalization of Ramanujan's result was started by Shevelev who proved that if apply Ramanujan's theorem to equation (1) under assumption $a+b+3=0$ we eventually get the following

Theorem B (Shevelev, 1988, [7], see also [8]). *Let $p,q,r \in \mathbb{R},\ r \neq 0$ be such that*

$$pr^{\frac{1}{3}} + 3r^{\frac{2}{3}} + q = 0$$

and let the polynomial x^3+px^2+qx+r have real roots x_1, x_2, x_3. Then

$$x_1^{\frac{1}{3}} + x_2^{\frac{1}{3}} + x_3^{\frac{1}{3}} = (-p + 6r^{\frac{1}{3}} + 3(9r - pq)^{\frac{1}{3}})^{\frac{1}{3}}$$

and

$$(x_1x_2)^{\frac{1}{3}} + (x_2x_3)^{\frac{1}{3}} + (x_3x_1)^{\frac{1}{3}} = (q + 6r^{\frac{2}{3}} - 3(9r^2 - pqr)^{\frac{1}{3}})^{\frac{1}{3}}.$$

This result gave birth to

Definition 1.1 (Shevelev, 2007, [8]). *Let $p,q,r \in \mathbb{R},\ r \neq 0$. A cubic polynomial*

$$\pi(x) = x^3 + px^2 + qx + r$$

is called a ***Ramanujan cubic polynomial (RCP)*** *if and only if it has real roots and satisfies the condition*

$$pr^{\frac{1}{3}} + 3r^{\frac{2}{3}} + q = 0. \tag{2}$$

Note that RCP is a monic polynomial (i.e. the leading coefficient equals 1).

As an example, we give a polynomial considered by Ramanujan, namely

$$(*) \qquad x^3 - 3x^2 - 6x + 8 = (x-1)(x+2)(x-4).$$

Note that roots of this RCP form a geometric sequence (see also formula (18), page 188, and considerations after)

There are known two representations of RCPs and their roots.

Theorem C (Wituła, 2010, [10]). *A monic cubic polynomial $\pi(x)$ is RCP if and only if there exist $r, \gamma \in \mathbb{R},\ r \neq 0, \gamma \neq 1, \gamma \neq 2$ such that*

$$\pi(x) = x^3 + \frac{P(\gamma-1)}{(\gamma-1)(2-\gamma)} r^{\frac{1}{3}} x^2 + \frac{P(2-\gamma)}{(\gamma-1)(2-\gamma)} r^{\frac{2}{3}} x + r, \quad P(\gamma) := \gamma^3 - 3\gamma + 1. \tag{3}$$

Moreover, roots of $\pi(x)$ are

$$\frac{1}{2-\gamma} r^{\frac{1}{3}}, \qquad (\gamma-1) r^{\frac{1}{3}} \qquad \text{and} \qquad \frac{2-\gamma}{1-\gamma} r^{\frac{1}{3}}. \tag{4}$$

Observe that polynomial $(*)$ is obtained for $\gamma = 3,\ r = 8$.

Theorem D (Barbero, Cerruti, Murru, Abrate, 2013, [1]). *A monic cubic polynomial $\pi(x)$ is RCP if and only if there exist $h, g \in \mathbb{R},\ g \neq 0$ such that*

$$\pi(x) = x^3 + hgx^2 - (h+3)g^2x + g^3. \tag{5}$$

Moreover, roots of $\pi(x)$ *depend on* h *and for* $\tau(h) := h^2 + 3h + 9$ *they are of the form*

$$-\frac{g}{3}\left(h + 2\sqrt{\tau(h)}\cos\left(\frac{1}{3}\left(\arctan\frac{3\sqrt{3}}{3+2h} + k\pi\right)\right)\right),\ k = 0,2,4,\ h > -\frac{3}{2},$$

$$-\frac{g}{3}\left(h - 2\sqrt{\tau(h)}\cos\left(\frac{1}{3}\left(\arctan\frac{3\sqrt{3}}{3+2h} + k\pi\right)\right)\right),\ k = 0,2,4,\ h < -\frac{3}{2},$$

$$\frac{g}{2},\ -g,\ 2g \quad \textit{for} \quad h = -\frac{3}{2}. \tag{6}$$

Observe that polynomial $(*)$ is obtained for $h = -\frac{3}{2},\ g = 2$.

We provide one more representation of RCPs (proven in Section 4), namely

Theorem 1.2. *A monic cubic polynomial* $\pi(x)$ *is RCP if and only if there exist* $r, s \in \mathbb{R},\ r \neq 0,\ s \leqslant \frac{9}{4}$ *such that*

$$\pi(x) = x^3 + \frac{-3 \mp \sqrt{9-4s}}{2} r^{\frac{1}{3}} x^2 + \frac{-3 \pm \sqrt{9-4s}}{2} r^{\frac{2}{3}} x + r. \tag{7}$$

Moreover, roots of $\pi(x)$ *are*

$$\begin{gathered}\frac{r^{\frac{1}{3}}}{6}\left(3 \pm \sqrt{9-4s} \pm 4\sqrt{9-s}\cos\left(\frac{1}{3}\arctan\sqrt{\frac{27}{9-4s}}\right)\right),\\ \frac{r^{\frac{1}{3}}}{6}\left(3 \pm \sqrt{9-4s} \mp 4\sqrt{9-s}\sin\left(\frac{\pi}{6} + \frac{1}{3}\arctan\sqrt{\frac{27}{9-4s}}\right)\right),\\ \frac{r^{\frac{1}{3}}}{6}\left(3 \pm \sqrt{9-4s} \mp 4\sqrt{9-s}\sin\left(\frac{\pi}{6} - \frac{1}{3}\arctan\sqrt{\frac{27}{9-4s}}\right)\right).\end{gathered} \tag{8}$$

with signs taken respectively to those in (7).

Observe that polynomial $(*)$ is obtained for $r = 8,\ s = \frac{9}{4}$.

1.2. Examination of RCPs

In spite of full characterizations of RCPs there are some results which help to understand their nature. Below we list some of them.

Notation. From now on, $\pi(x) := x^3 + px^2 + qx + r$ denotes RCP with roots x_1, x_2, x_3. Moreover, $h, g, \gamma, P(\gamma), \tau(h)$ are as in representations of RCPs (see Theorems C and D) and

$$s := \frac{pq}{r} \tag{9}$$

will be called the **Shevelev parameter**.

New RCPs on a base of $\pi(x)$:

P.1. [8] For every $a \in \mathbb{R},\ a \neq 0$ the polynomial $x^3+apx^2+a^2qx+a^3r$ is also RCP with roots ax_1, ax_2, ax_3.

P.2. [8] The polynomial $x^3+qx^2+prx+r^2$ is also RCP with roots $\frac{r}{x_1}, \frac{r}{x_2}, \frac{r}{x_3}$.

Properties of roots of RCPs:

P.3. Each RCP has 3 distinct roots (follows from representations, see also Section 2).

P.4. [1] $\{x_1,x_2,x_3\}=\left\{x_1, \frac{g^2}{g-x_1}, \frac{g(g-x_1)}{x_1}\right\}=\left\{x_1 r^{-\frac{1}{3}},\ -\frac{1}{x_1+1}r^{-\frac{1}{3}},\ -\frac{x_1+1}{x_1}r^{-\frac{1}{3}}\right\}$.

P.5. [8] $\left\{\frac{r^{\frac{2}{3}}}{x_1}, \frac{r^{\frac{2}{3}}}{x_2}, \frac{r^{\frac{2}{3}}}{x_3}\right\}=\left\{r^{\frac{1}{3}}-x_1, r^{\frac{1}{3}}-x_2, r^{\frac{1}{3}}-x_3\right\}$.

P.6. [8]

$$x_1^{\frac{1}{3}}+x_2^{\frac{1}{3}}+x_3^{\frac{1}{3}}=(-p+6r^{\frac{1}{3}}+3(9r-pq)^{\frac{1}{3}})^{\frac{1}{3}}, \tag{10}$$

$$(x_1x_2)^{\frac{1}{3}}+(x_2x_3)^{\frac{1}{3}}+(x_3x_1)^{\frac{1}{3}}=(q+6r^{\frac{2}{3}}-3(9r^2-pqr)^{\frac{1}{3}})^{\frac{1}{3}}, \tag{11}$$

$$x_1^{-\frac{1}{3}}+x_2^{-\frac{1}{3}}+x_3^{-\frac{1}{3}}=r^{-\frac{1}{3}}(-q-6r^{\frac{2}{3}}+3(9r^2-pqr)^{\frac{1}{3}})^{\frac{1}{3}}, \tag{12}$$

$$\left(\frac{x_1}{x_2}\right)^{\frac{1}{3}}+\left(\frac{x_2}{x_1}\right)^{\frac{1}{3}}+\left(\frac{x_1}{x_3}\right)^{\frac{1}{3}}+\left(\frac{x_3}{x_1}\right)^{\frac{1}{3}}+\left(\frac{x_2}{x_3}\right)^{\frac{1}{3}}+\left(\frac{x_3}{x_2}\right)^{\frac{1}{3}}=\left(\frac{pq}{r}-9\right)^{\frac{1}{3}}. \tag{13}$$

P.7. [7]

$$\begin{aligned}&\left(\frac{x_1}{x_2}\right)^{\frac{1}{3}}+\left(\frac{x_2}{x_1}\right)^{\frac{1}{3}}+\left(\frac{x_1}{x_3}\right)^{\frac{1}{3}}+\left(\frac{x_3}{x_1}\right)^{\frac{1}{3}}+\left(\frac{x_2}{x_3}\right)^{\frac{1}{3}}+\left(\frac{x_3}{x_2}\right)^{\frac{1}{3}}=\\&=\left(\frac{x_1}{x_2}+\frac{x_2}{x_1}+\frac{x_1}{x_3}+\frac{x_3}{x_1}+\frac{x_2}{x_3}+\frac{x_3}{x_2}-6\right)^{\frac{1}{3}}.\end{aligned} \tag{14}$$

Properties of RCPs connected to the Shevelev parameter:

P.8. [8] $s \leqslant \frac{9}{4}$.

P.9. [8] (see also Theorem 4.1, page 195) If two RCPs with roots y_1, y_2, y_3 and z_1, z_2, z_3 have the same Shevelev parameter then

$$\left\{\frac{y_1}{y_2}, \frac{y_2}{y_1}, \frac{y_1}{y_3}, \frac{y_3}{y_1}, \frac{y_2}{y_3}, \frac{y_3}{y_2}\right\}=\left\{\frac{z_1}{z_2}, \frac{z_2}{z_1}, \frac{z_1}{z_3}, \frac{z_3}{z_1}, \frac{z_2}{z_3}, \frac{z_3}{z_2}\right\}.$$

P.10. [10] $s=9-\frac{(\gamma-1)(\gamma-2)+1)^3}{((\gamma-1)(\gamma-2))^2}=\frac{P(\gamma-1)P(2-\gamma)}{((\gamma-1)(2-\gamma))^2}$.

P.11. (Theorem 4.1) For each real number $a<\frac{9}{4}$ there are exactly two distinct families of RCPs (depending on r only) with the Shevelev parameter equal to a and only one family for the limit value $\frac{9}{4}$.

1.3. Identities

Properties of roots of Ramanujan cubic polynomials are a source of nontrivial and beautiful algebraic identities. Below we list some of them in the chronological order.

1. Berndt, Chan, Zhang [4]

$$\left((a^2-7a+1)+(6a-3)a^{\frac{1}{3}}+(6a-3)a^{\frac{2}{3}}\right)^{\frac{1}{3}}=1+a^{\frac{1}{3}}+a^{\frac{2}{3}},\ a\in\mathbb{R}.$$

Note that the Ramanujan's identity (Q682) is obtained for $a=2$.

2. Shevelev, [8]

$$\left(\frac{\cos\frac{2\pi}{n}}{\cos\frac{4\pi}{n}}\right)^{\frac{1}{3}}+\left(\frac{\cos\frac{4\pi}{n}}{\cos\frac{2\pi}{n}}\right)^{\frac{1}{3}}+\left(\frac{\cos\frac{2\pi}{n}}{\cos\frac{8\pi}{n}}\right)^{\frac{1}{3}}+\left(\frac{\cos\frac{8\pi}{n}}{\cos\frac{2\pi}{n}}\right)^{\frac{1}{3}}+\left(\frac{\cos\frac{4\pi}{n}}{\cos\frac{8\pi}{n}}\right)^{\frac{1}{3}}+\left(\frac{\cos\frac{8\pi}{n}}{\cos\frac{4\pi}{n}}\right)^{\frac{1}{3}}=$$
$$=-n^{\frac{1}{3}},\qquad n=7,9,$$

$$\left(\frac{\sin\frac{2\pi}{7}}{\sin\frac{4\pi}{7}}\right)^2\left(\frac{\cos\frac{4\pi}{7}}{\cos\frac{8\pi}{7}}\right)^{\frac{1}{3}}+\left(\frac{\sin\frac{4\pi}{7}}{\sin\frac{2\pi}{7}}\right)^2\left(\frac{\cos\frac{8\pi}{7}}{\cos\frac{4\pi}{7}}\right)^{\frac{1}{3}}+\left(\frac{\sin\frac{2\pi}{7}}{\sin\frac{8\pi}{7}}\right)^2\left(\frac{\cos\frac{4\pi}{7}}{\cos\frac{2\pi}{7}}\right)^{\frac{1}{3}}+$$
$$+\left(\frac{\sin\frac{8\pi}{7}}{\sin\frac{2\pi}{7}}\right)^2\left(\frac{\cos\frac{2\pi}{7}}{\cos\frac{4\pi}{7}}\right)^{\frac{1}{3}}+\left(\frac{\sin\frac{4\pi}{7}}{\sin\frac{8\pi}{7}}\right)^2\left(\frac{\cos\frac{8\pi}{7}}{\cos\frac{2\pi}{7}}\right)^{\frac{1}{3}}+\left(\frac{\sin\frac{8\pi}{7}}{\sin\frac{4\pi}{7}}\right)^2\left(\frac{\cos\frac{2\pi}{7}}{\cos\frac{8\pi}{7}}\right)^{\frac{1}{3}}=-3\cdot 7^{\frac{1}{3}}.$$

3. Wituła, Słota, [13]

$$\sin^2\frac{2\pi}{7}\left(2\cos\frac{4\pi}{7}\right)^{\frac{1}{3}}+\sin^2\frac{4\pi}{7}\left(2\cos\frac{8\pi}{7}\right)^{\frac{1}{3}}+\sin^2\frac{8\pi}{7}\left(2\cos\frac{8\pi}{7}\right)^{\frac{1}{3}}=$$
$$=-\frac{1}{4}\left(63(1+7^{\frac{1}{3}})\right)^{\frac{1}{3}}.$$

4. Wituła, [9]

$$\left(\frac{\cos\frac{2\pi}{7}}{\cos\frac{4\pi}{7}}\right)^{\frac{1}{3}}\left(2\cos\frac{2\pi}{7}\right)^k+\left(\frac{\cos\frac{4\pi}{7}}{\cos\frac{8\pi}{7}}\right)^{\frac{1}{3}}\left(2\cos\frac{4\pi}{7}\right)^k+\left(\frac{\cos\frac{8\pi}{7}}{\cos\frac{2\pi}{7}}\right)^{\frac{1}{3}}\left(2\cos\frac{8\pi}{7}\right)^k=$$
$$=7^{\frac{1}{3}}\psi_k,$$
$$\psi_0=-1,\ \psi_1=0,\ \psi_2=-3,\quad \psi_{k+3}+\psi_{k+2}-2\psi_{k+1}-\psi_k=0,\ k\in\mathbb{Z},$$
$$\left(\frac{\cos\frac{2\pi}{7}}{\cos\frac{8\pi}{7}}\right)^{\frac{1}{3}}\left(2\cos\frac{2\pi}{7}\right)^k+\left(\frac{\cos\frac{4\pi}{7}}{\cos\frac{2\pi}{7}}\right)^{\frac{1}{3}}\left(2\cos\frac{4\pi}{7}\right)^k+\left(\frac{\cos\frac{8\pi}{7}}{\cos\frac{4\pi}{7}}\right)^{\frac{1}{3}}\left(2\cos\frac{8\pi}{7}\right)^k=$$
$$=49^{\frac{1}{3}}\phi_k,$$
$$\phi_0=0,\ \phi_1=-1,\ \phi_2=1,\quad \phi_{k+3}+\phi_{k+2}-2\phi_{k+1}-\phi_k=0,\ k\in\mathbb{Z}.$$

5. Wituła, [12]

$$\left(\frac{\cos\frac{2\pi}{9}}{\cos\frac{4\pi}{9}}\right)^{\frac{1}{3}}\left(2\cos\frac{2\pi}{9}\right)^k+\left(\frac{\cos\frac{4\pi}{9}}{\cos\frac{8\pi}{9}}\right)^{\frac{1}{3}}\left(2\cos\frac{4\pi}{9}\right)^k+\left(\frac{\cos\frac{8\pi}{9}}{\cos\frac{2\pi}{9}}\right)^{\frac{1}{3}}\left(2\cos\frac{2\pi}{9}\right)^k=$$
$$=3^{\frac{1}{3}}\Psi_k,$$
$$\Psi_0=0,\ \Psi_1=3,\ \Psi_2=0,\quad \Psi_{k+3}-3\Psi_{k+1}+\Psi_k=0,\ k\in\mathbb{Z},$$

$$\left(\frac{\cos\frac{2\pi}{9}}{\cos\frac{8\pi}{9}}\right)^{\frac{1}{3}}\left(2\cos\frac{2\pi}{9}\right)^{k}+\left(\frac{\cos\frac{4\pi}{9}}{\cos\frac{2\pi}{9}}\right)^{\frac{1}{3}}\left(2\cos\frac{4\pi}{9}\right)^{k}+\left(\frac{\cos\frac{8\pi}{9}}{\cos\frac{4\pi}{9}}\right)^{\frac{1}{3}}\left(2\cos\frac{2\pi}{9}\right)^{k}= \\ = 9^{\frac{1}{3}}\varPhi_k,$$

$$\varPhi_0-1,\ \varPhi_1=1,\ \varPhi_2=-4,\quad \varPhi_{k+3}-3\varPhi_{k+1}+\varPhi_k=0,\ k\in\mathbb{Z}.$$

6. Barbero, Cerruti, Murru, Abrate, [1]

$$\sqrt{7}\cos\left(\frac{1}{3}\arctan\frac{9\sqrt{3}}{10}\right)-\sqrt{21}\sin\left(\frac{1}{3}\arctan\frac{9\sqrt{3}}{10}\right)=1,$$

and the last identity in the list, which is of a different nature, but still nice:

$$\frac{1}{(\pi^3-1)^{\frac{1}{3}}}-\frac{(\pi^3-1)^{\frac{1}{3}}}{\pi}+\left(\frac{3(\pi^6-\pi^3+1)}{\pi^2(\pi^3-1)^{\frac{2}{3}}}+\frac{\pi^9-6\pi^6+3\pi^3+1}{\pi^3(\pi^3-1)}\right)^{\frac{1}{3}}=\pi.$$

1.4. Families of cubic polynomials associated to RCPs

In a spirit of and in a connection with RCPs some new families of cubic polynomials were considered.

Definition 1.3 (Wituła, 2010, [11]). *A polynomial $\pi(x)=x^3+px^2+qx+r,\ p,q,r\in\mathbb{R}$, $r\neq 0$ is called*[1] *a* ***Ramanujan cubic polynomial of the second kind (RCP2)*** *if and only if*

$$q^3+p^3r+27r^2=0. \tag{15}$$

Note that each summand in the above condition is a cube of a summand in the definition of RCP. It is known that each RCP2 has the form

$$x^3+3\sqrt[3]{kr}x^2-3\sqrt[3]{(k+1)r^2}x+r, \tag{16}$$

where $k,r\in\mathbb{R},\ r\neq 0$ and that RCP2 is RCP if and only if $pqr=0$ [11]. Therefore, although many properties of RCPs and RCP2s are different, they have also many in common. We discuss some of them in this paper.

Definition 1.4 (Barbero, Cerruti, Murru, Abrate, 2013, [1]). *A polynomial $\pi(x)$ is called a* ***Shanks cubic polynomial (SCP)*** *if and only if there exists $h\in\mathbb{R}$ such that*

$$\pi(x)=x^3-hx^2-(h+3)x-1. \tag{17}$$

In view of Theorem D (page 183), every SCP is RCP for $g=-1$. Examination of SCPs allowed, among the others, to set up the formulas and properties of roots of RCPs via the Galois theory.

For properties of the above mentioned classes of cubic polynomials see the papers cited in definitions.

[1] Although the case $r=0$ was not formally excluded from the original definition of RCP2, it was assumed to be.

1.5. Organization of a paper

The rest of this paper is organized as follows: in Section 2 we discuss properties of roots of RCPs and RCP2s. In Section 3 we show how to generate RCP2s on a base of RCPs in a spirit of Ramanujan's approach, namely by taking cubic roots. In Section 4 we prove the form of a new representation of RCPs (Theorem 1.2 above). Moreover, we show that this representation is actually given in terms of the Shevelev parameter $\frac{pq}{r}$, and therefore we can refine a statement of Theorem 4 from [10]. Section 5 is devoted to some asymptotic studies of RCP2s.

2. Differences and similarities for RCPs and RCP2s

In [11] it was proven that polynomials

$$x^3 + \sqrt[3]{p}x^2 + \frac{\sqrt{5}-1}{6}\sqrt[3]{p^2}x + \frac{1-\sqrt{5}}{54}p,$$

where $p \in \mathbb{R},\ p \neq 0$, are the only RCP2s that possess a double root

$$x_1 = x_2 = \frac{\sqrt[3]{p}}{3}\left(-1 - \sqrt[3]{\sqrt{5}-2}\right).$$

The third root is of the form

$$x_3 = \frac{\sqrt[3]{p}}{3}\left(-1 + 2\sqrt[3]{\sqrt{5}-2}\right).$$

Also note that from (22) in [11] we get

$$1 + \left(\frac{x_1}{x_3}\right)^{\frac{1}{3}} + \left(\frac{x_3}{x_1}\right)^{\frac{1}{3}} = \left(\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{1}{3}} - \left(\frac{\sqrt{5}+1}{2}\right)^{\frac{1}{3}}\right) = \left(\left(\frac{1}{\varphi}\right)^{\frac{1}{3}} - (\varphi)^{\frac{1}{3}}\right)^{\frac{1}{3}},$$

where $\varphi = \frac{\sqrt{5}+1}{2}$ is the golden ratio.

However, there does not exist RCP with double root. Indeed, if we assume, contrary to our claim, that RCP possesses a double root, then from the formulas for roots presented in Theorem C (page 183) we get

$$\frac{1}{2-\gamma} = \gamma - 1 \quad \text{or} \quad \frac{1}{2-\gamma} = \frac{2-\gamma}{1-\gamma} \quad \text{or} \quad \gamma - 1 = \frac{2-\gamma}{1-\gamma}.$$

Each equation simplifies to $\gamma^2 - 3\gamma + 3 = 0$ whence γ cannot be a real number, a contradiction.

Next, polynomials of the form

$$x^3 - \frac{3}{2}r^{\frac{1}{3}}x^2 - \frac{3}{2}r^{\frac{2}{3}}x + r = (x - 2\sqrt[3]{r})(x + \sqrt[3]{r})\left(x - \frac{1}{2}\sqrt[3]{r}\right) \tag{18}$$

are the only RCPs roots of which form a geometric sequence. Verification is straightforward – it is enough to consider 6 cases using the form of roots given in Theorem C on page 183 (3 of them are impossible and in the remaining 3 we get polynomials (18)).

However, there does not exist RCP2 with this property. Indeed, if we assume, contrary to our claim, that some RCP2 has roots of the form aq, aq^2, aq^3 then from (16) and Viete's formulas we get the following system

$$\begin{cases} aq^2(\frac{1}{q}+1+q) = -3\sqrt[3]{kr} \\ a^2q^4(\frac{1}{q}+1+q) = -3\sqrt[3]{(k+1)r^2} \\ a^3q^6 = -r \end{cases}$$

From the last equation we get $aq^2 = -\sqrt[3]{r}$ and substituting that to the first and second equations yields contradiction. That also means that the statement of Theorem 7(g) in [11] is satisfied trivially.

So the above two properties of roots differ the two considered families of polynomials. The following property shows their similarity.

Theorem 2.1. *If $\pi(x) = x^3+px^2+qx+r$ is RCP (resp. RCP2) with roots x_1, x_2, x_3, then $\pi_{inv}(x) := x^3 + \frac{q}{r}x^2 + \frac{p}{r}x + \frac{1}{r}$ is also RCP (resp. RCP2) with roots $\frac{1}{x_1}, \frac{1}{x_2}, \frac{1}{x_3}$.*

Proof. First observe that by assumption and Viete's formulas we have the following equalities $p = -(x_1+x_2+x_3),\ q = x_1x_2+x_1x_3+x_2x_3,\ r = -x_1x_2x_3$ whence

$$\pi_{inv}(x) = \left(x-\frac{1}{x_1}\right)\left(x-\frac{1}{x_2}\right)\left(x-\frac{1}{x_3}\right),$$

so $\pi_{inv}(x)$ has a desired form of roots. Next, by Definition 1.1 (page 183) for $\pi(x)$ we have $pr^{\frac{1}{3}} + 3r^{\frac{2}{3}} + q = 0$, whence for $\pi_{inv}(x)$ we get

$$\frac{q}{r}\frac{1}{r^{\frac{1}{3}}} + 3\frac{1}{r^{\frac{2}{3}}} + \frac{p}{r} = \frac{1}{r^{\frac{4}{3}}}\left(q + 3r^{\frac{2}{3}} + pr^{\frac{1}{3}}\right) = 0,$$

which means that $\pi_{inv}(x)$ is also RCP.

Now, let $\pi(x)$ be RCP2. Then by Definition 1.3 (page 187) we get

$$p^3r + 27r^2 + q^3 = 0$$

and hence

$$\left(\frac{q}{r}\right)^3\frac{1}{r} + 27\frac{1}{r^2} + \left(\frac{p}{r}\right)^3 = \frac{1}{r^4}\left(q^3+27r^2+p^3r\right) = 0,$$

which means that $\pi_{inv}(x)$ is also RCP2, as required. □

Remark 2.2. The part of the proof for RCPs also follows if we apply Property P.2 and then P.1 for $a = \frac{1}{r}$.

3. How RCP generates RCP2

In [11] it was examined what is a connection between RCPs and RCP2s and what are the conditions on some parameters connected with a polynomial to be RCP or RCP2 (conclusions were given in Theorem 7). In the previous section we generated RCPs (resp. RCP2s) on a base of other RCPs (resp. RCP2s) in a spirit of Shevelev, namely considering inverses of roots. Now we show how to generate RCP2 on a base of some RCP in a spirit of Ramanujan, by considering cubic roots of roots.

Theorem 3.1. *If a polynomial*

$$\pi(x) = (x - x_1)(x - x_2)(x - x_3) = x^3 + px^2 + qx + r$$

is RCP then the polynomial

$$\sqrt[3]{\pi}(x) := \left(x - \sqrt[3]{x_1}\right)\left(x - \sqrt[3]{x_2}\right)\left(x - \sqrt[3]{x_3}\right)$$

is RCP2 if and only if

$$\begin{cases} pr^{\frac{1}{3}} + 3r^{\frac{2}{3}} + q = 0 \\ r = -\frac{pq}{207} \end{cases} \tag{19}$$

or, equivalently, if and only if there exists $a \in \mathbb{R}$, $a \neq 0$ *such that*

$$\begin{gathered} p = 9a, \quad q = 23\left(\frac{6}{1+\sqrt{93}}\right)^3 a^2, \quad r = -\left(\frac{6}{1+\sqrt{93}}\right)^3 a^3, \\ \text{or} \\ p = 9a, \quad q = 23\left(\frac{6}{1-\sqrt{93}}\right)^3 a^2, \quad r = -\left(\frac{6}{1-\sqrt{93}}\right)^3 a^3. \end{gathered} \tag{20}$$

Moreover, in this case we have

$$\begin{aligned} \frac{x_1}{a} &= -3 - \frac{6\sqrt{6}}{23}(\sqrt{93} \mp 1)\cos\left(\frac{1}{3}\operatorname{arccot}\sqrt{31}\right), \\ \frac{x_2}{a} &= -3 + \frac{6\sqrt{6}}{23}(\sqrt{93} \mp 1)\sin\left(\frac{\pi}{6} + \frac{1}{3}\operatorname{arccot}\sqrt{31}\right), \\ \frac{x_3}{a} &= -3 + \frac{6\sqrt{6}}{23}(\sqrt{93} \mp 1)\sin\left(\frac{\pi}{6} - \frac{1}{3}\operatorname{arccot}\sqrt{31}\right), \end{aligned} \tag{21}$$

with upper and lower signs taken respectively to those in (20) and therefore

$$\begin{aligned} \sqrt[3]{\pi}(x) = x^3 &+ \sqrt[3]{\frac{9\left(9 \pm \sqrt{93}\right)}{1 \pm \sqrt{93}}}ax^2 \\ &+ \frac{6}{1 \pm \sqrt{93}}\sqrt[3]{\left(21 \mp 2\sqrt{93}\right)a^2}x - \frac{6}{1 \pm \sqrt{93}}a. \end{aligned} \tag{22}$$

Proof. The first equation in (19) follows from the fact that $\pi(x)$ is RCP. Next, using Shevelev's identities (see Property P.6, page 185) and Viete's formulas we obtain

$$\sqrt[3]{\pi}(x) = x^3 - \sqrt[3]{-p - 6\sqrt[3]{r} + 3\sqrt[3]{9r - pq}}x^2 + \\ + \sqrt[3]{q + 6\sqrt[3]{r^2} - 3\sqrt[3]{9r^2 - pqr}}x + \sqrt[3]{r}.$$

So by Definition 1.3 this polynomial is RCP2 if and only if

$$\left(q + 6\sqrt[3]{r^2} - 3\sqrt[3]{9r^2 - pqr}\right) + \left(p\sqrt[3]{r} + 6\sqrt[3]{r^2} - 3\sqrt[3]{9r^2 - pqr}\right) + 27\sqrt[3]{r^2} = 0,$$

whence

$$36\sqrt[3]{r^2} - 6\sqrt[3]{9r^2 - pqr} = 0 \iff 6^3 r^2 = 9r^2 - pqr \iff -pqr = (6^3 - 9)r^2,$$

and finally, since $r \neq 0$, we get

$$r = -\frac{pq}{207},$$

which is the second equation in the system. To prove (20) we substitute the second equation in (19) to the first one obtaining

$$-p\sqrt[3]{\frac{pq}{207}} + 3\sqrt[3]{\left(\frac{pq}{207}\right)^2} + q = 0,$$

and if we set $p := 9a,\ q := 23c$, then $a \neq 0,\ c \neq 0$ and we get

$$-9a\sqrt[3]{ac} + 3\sqrt[3]{(ac)^2} + 23b = 0 \iff -3\frac{a^{\frac{4}{3}}}{c^{\frac{2}{3}}} + \frac{a^{\frac{2}{3}}}{c^{\frac{1}{3}}} + \frac{23}{3} = 0,$$

which gives

$$\frac{a^{\frac{2}{3}}}{c^{\frac{1}{3}}} = \frac{1 \pm \sqrt{93}}{6},$$

whence

$$q = 23c = 23\left(\frac{6}{1 \pm \sqrt{93}}\right)^3 a^2.$$

Substituting for p, q in the second equation in (19) we get $r = -\left(\frac{6}{1 \pm \sqrt{93}}\right)^3 a^3$ which means that (20) holds.

Now, we take $\pi(x)$ with p, q, r given in (20) under assumption $a = 1$. To obtain a depressed polynomial we take $z := x - 3$ and hence it has the form

$$z^3 - \frac{2592}{(\sqrt{93} \pm 1)^2}z + \frac{5184\sqrt{93}}{(\sqrt{93} \pm 1)^3}.$$

Now we shall use a refined version of Cardano's formulas given in [14]. It was proven that roots of the polynomial $z^3 + Az + B$ for which $\varDelta := \left(\frac{B}{2}\right)^2 + \left(\frac{A}{3}\right)^3 < 0$ and $B > 0$ (which is exactly our case) are of the following form

$$
\begin{gathered}
-2\sqrt{-\frac{A}{3}}\cos\left(\frac{1}{3}\arctan\sqrt{-\left(\frac{2}{B}\right)^2\Delta}\right),\\
2\sqrt{-\frac{A}{3}}\sin\left(\frac{\pi}{6}\pm\frac{1}{3}\arctan\sqrt{-\left(\frac{2}{B}\right)^2\Delta}\right).
\end{gathered}
\tag{23}
$$

Since $\arctan q = \operatorname{arccot}\frac{1}{q}$ for every positive real q, the above formulas yield roots of $\pi(x)$ for $a = 1$ which are exactly expressions on right sides in (21). Then the final formulas follows from Property P.1.

Finally, (22) is obtained by straightforward calculations. □

From the above theorem some nontrivial identities can be derived. First of all, for $a = \frac{1}{9}(1 \pm \sqrt{93})b$ and $M := \frac{1}{3}\sqrt[3]{b}\sqrt[6]{6\left(47 \mp 3\sqrt{93}\right)}$, from (22) in [11] we get the new rescaled $\sqrt[3]{\pi}(x)$ of the form

$$
\begin{aligned}
\sqrt[3]{\pi}(x) &= x^3 + \sqrt[3]{\left(9\pm\sqrt{93}\right)b}\,x^2 - \sqrt[3]{\frac{2}{3}\left(9\mp\sqrt{93}\right)b^2}\,x - \frac{2}{3}b = \\
&= \left(x \pm M\left(\sqrt{\frac{1}{14}(11\pm\sqrt{93})} + 2\cos\left(\frac{1}{3}\arctan\frac{\sqrt{3}(\sqrt{93}\pm 9)}{2}\right)\right)\right)\times\\
&\quad\times\left(x \pm M\left(\sqrt{\frac{1}{14}(11\pm\sqrt{93})} - 2\sin\left(\frac{\pi}{6}+\frac{1}{3}\arctan\frac{\sqrt{3}(\sqrt{93}\pm 9)}{2}\right)\right)\right)\times\\
&\quad\times\left(x \pm M\left(\sqrt{\frac{1}{14}(11\pm\sqrt{93})} - 2\sin\left(\frac{\pi}{6}-\frac{1}{3}\arctan\frac{\sqrt{3}(\sqrt{93}\pm 9)}{2}\right)\right)\right) = \\
&= \left(x \pm \left(\frac{1}{3}\sqrt[3]{(\sqrt{93}\pm 9)b} + 2M\cos\left(\frac{1}{3}\arctan\frac{\sqrt{3}(\sqrt{93}\pm 9)}{2}\right)\right)\right)\times\\
&\quad\times\left(x \pm \left(\frac{1}{3}\sqrt[3]{(\sqrt{93}\pm 9)b} - 2M\sin\left(\frac{\pi}{6}+\frac{1}{3}\arctan\frac{\sqrt{3}(\sqrt{93}\pm 9)}{2}\right)\right)\right)\times\\
&\quad\times\left(x \pm \left(\frac{1}{3}\sqrt[3]{(\sqrt{93}\pm 9)b} - 2M\sin\left(\frac{\pi}{6}-\frac{1}{3}\arctan\frac{\sqrt{3}(\sqrt{93}\pm 9)}{2}\right)\right)\right),
\end{aligned}
\tag{24}
$$

where all roots taken are the real ones. Moreover, we obtain

$$
M\sqrt{\frac{1}{14}(11\pm\sqrt{93})} = \frac{1}{3}\sqrt[3]{(\sqrt{93}\pm 9)b},
$$

which implies equalities

$$
\sqrt[6]{\frac{1}{4}(47\mp 3\sqrt{93})}\sqrt{\frac{1}{7}(11\pm\sqrt{93})} = \sqrt[3]{\sqrt{31}\pm 3\sqrt{3}}.
$$

Next, while looking for roots of the polynomial $\sqrt[3]{\pi}(x)$, inspired by suggestions given by *Mathematica*, we deduced the following equalities

$$\sqrt[3]{6(47 \pm 3\sqrt{93})} = \sqrt[3]{(\sqrt{93} \mp 9)^2} \pm \sqrt[3]{18(\sqrt{93} \pm 9)}.$$

Furthermore, we obtained two Ramanujan type identities for RCP2s, involving 9-th roots of RCPs (with signs taken respectively to those in formulas (20))

$$\begin{aligned}
&\left(\frac{\sqrt{93} \pm 1}{2 \cdot 3^4}\right)^{\frac{1}{9}} \left(x_1^{\frac{1}{9}} + x_2^{\frac{1}{9}} + x_3^{\frac{1}{9}}\right) = \\
&= \left(\pm 2 - \left(\frac{\sqrt{93} \pm 9}{2 \cdot 3^2}\right)^{\frac{1}{3}} \mp \left(\frac{3^4}{2}\right)^{\frac{1}{9}} \left(\left(\sqrt[3]{\sqrt{93}+9} - \sqrt[3]{\sqrt{93}-9}\right)^{\frac{1}{3}} + \right.\right. \\
&\quad \left.\left. + \left(\sqrt[3]{\sqrt{93}+9} - \sqrt[3]{\sqrt{93}-9} - \sqrt[3]{\frac{2^4}{3}}\right)^{\frac{1}{3}}\right)\right)^{\frac{1}{3}},
\end{aligned} \tag{25}$$

and finally we get an analogue of Shevelev identity for RCP2s, involving 9-th roots of RCPs (which is the same in both cases), namely

$$\begin{aligned}
&\left(\frac{2}{3^4}\right)^{\frac{1}{9}} \left(\left(\frac{x_1}{x_2}\right)^{\frac{1}{9}} + \left(\frac{x_2}{x_1}\right)^{\frac{1}{9}} + \left(\frac{x_1}{x_3}\right)^{\frac{1}{9}} + \left(\frac{x_3}{x_1}\right)^{\frac{1}{9}} + \left(\frac{x_2}{x_3}\right)^{\frac{1}{9}} + \left(\frac{x_3}{x_2}\right)^{\frac{1}{9}}\right) = \\
&= \left(\sqrt[3]{\sqrt{93}-9} - \sqrt[3]{\sqrt{93}+9} + \sqrt[3]{\frac{2^4}{3}}\right)^{\frac{1}{3}} - \left(\sqrt[3]{\sqrt{93}+9} - \sqrt[3]{\sqrt{93}-9}\right)^{\frac{1}{3}}.
\end{aligned} \tag{26}$$

Additionally, for the upper signs in (25) we deduce the following relation

$$\begin{aligned}
&-\left(x_1^{\frac{1}{9}} + x_2^{\frac{1}{9}} + x_3^{\frac{1}{9}}\right)\left(21 + 2\sqrt{93}\right)^{\frac{1}{9}} = \\
&\quad = \left(1 + \sqrt{2(11+\sqrt{93})} \cos\left(\frac{1}{3} \arctan \frac{\sqrt{3}}{2}(\sqrt{93}-9)\right)\right)^{\frac{1}{3}} + \\
&\quad + \left(1 - \sqrt{2(11+\sqrt{93})} \sin\left(\frac{\pi}{6} - \frac{1}{3} \arctan \frac{\sqrt{3}}{2}(\sqrt{93}-9)\right)\right)^{\frac{1}{3}} + \\
&\quad + \left(1 - \sqrt{2(11+\sqrt{93})} \sin\left(\frac{\pi}{6} + \frac{1}{3} \arctan \frac{\sqrt{3}}{2}(\sqrt{93}-9)\right)\right)^{\frac{1}{3}}.
\end{aligned} \tag{27}$$

4. Representations of RCPs

Proof of Theorem 1.2. Assume first that $\pi(x) = x^3 + px^2 + qx + r$ is RCP such that the Shevelev parameter $s = \frac{pq}{r} \neq 0$.

Since $pr^{\frac{1}{3}} + 3r^{\frac{2}{3}} + q = 0$, we get

$$p\frac{(pq)^{\frac{1}{3}}}{s^{\frac{1}{3}}} + 3\frac{(pq)^{\frac{2}{3}}}{s^{\frac{2}{3}}} + q = 0,$$

from which, multiplying by $s^{\frac{2}{3}}/q^{\frac{1}{3}}$, we obtain

$$s\left(\frac{p^2}{s}\right)^{\frac{2}{3}} + 3\left(\frac{p^2}{s}\right)^{\frac{1}{3}}(sq)^{\frac{1}{3}} + (sq)^{\frac{2}{3}} = 0.$$

Now, under substitutions $z := (sq)^{\frac{1}{3}},\ y := \left(\frac{p^2}{s}\right)^{\frac{1}{3}}$, we get

$$z^2 + 3zy + sy^2 = 0 \iff z = \frac{-3 \pm \sqrt{9-4s}}{2}y.$$

Next, $s \leqslant \frac{9}{4}$ by Property P.8, which gives the desired bound for this parameter. Hence $9 - 4s \geqslant 0$, so from above we obtain

$$(s^2q)^{\frac{1}{3}} = \left(\frac{p^2}{r^2}\right)^{\frac{1}{3}} q = \frac{-3 \pm \sqrt{9-4s}}{2}(p^2)^{\frac{1}{3}},$$

whence

$$q = \frac{-3 \pm \sqrt{9-4s}}{2}r^{\frac{2}{3}}, \qquad p = \frac{sr}{q} = \frac{-3 \mp \sqrt{9-4s}}{2}r^{\frac{1}{3}}.$$

and therefore we get the formula (7), that is

$$\pi(x) = x^3 + \frac{-3 \mp \sqrt{9-4s}}{2}r^{\frac{1}{3}}x^2 + \frac{-3 \pm \sqrt{9-4s}}{2}r^{\frac{2}{3}}x + r$$

as required.

For $s = 0$ the calculations are straightforward, on a base of Theorem C (page 183). Namely, $s = 0$ if and only if either $P(2-\gamma) = 0$ or $P(\gamma-1) = 0$. The first equality takes place for $\gamma = 2\cos\frac{2^k\pi}{9} - 2,\ k = 1, 2, 3$. For all three values we obtain $\frac{P(\gamma-1)}{(\gamma-1)(2-\gamma)} = -3$ whence $\pi(x) = x^3 - 3r^{\frac{1}{3}}x^2 + r$ which suits the above formula for upper signs. Similarly, from the second equality we get (7) for lower signs. Thus (7) is valid for every $s \leqslant \frac{9}{4}$.

Finally, from Cardano-type formulas (23) applied to (7) we obtain roots of $\pi(x)$ defined in (8), which finishes the proof. $\square$

As an immediate consequence we get the following

Theorem 4.1. *1. A cubic polynomial* $\pi(x) = x^3 + px^2 + qx + r,\ p, q, r \in \mathbb{R},\ r \neq 0$ *is RCP if and only if it is of the form*

$$\pi_{1/2}(x; r) := x^3 + \frac{-3 \pm \sqrt{9-4s}}{2} r^{\frac{1}{3}} x^2 + \frac{-3 \mp \sqrt{9-4s}}{2} r^{\frac{2}{3}} x + r, \quad s = \frac{pq}{r}.$$

2. For each real number $a < \frac{9}{4}$ *there are exactly 2 distinct families of RCPs of the form* $\pi_1(x; r)$ *and* $\pi_2(x; r)$ *(depending on* r *only) with the Shevelev parameter equal to* a *and only one family for the limit value* $\frac{9}{4}$.

3. We have $\pi_1(x; 1) = \pi_2\left(\frac{1}{x}; 1\right)$*, whence roots* $x_1,\ x_2,\ x_3$ *of* $\pi_1(x; 1)$ *are reciprocals of roots of* $\pi_2(x; 1)$*. Hence the roots of* $\pi_1(x; r)$ *are* $r^{\frac{1}{3}} x_1,\ r^{\frac{1}{3}} x_2,\ r^{\frac{1}{3}} x_3$ *whereas roots of* $\pi_2(x; r)$ *are* $\frac{r^{\frac{1}{3}}}{x_1},\ \frac{r^{\frac{1}{3}}}{x_2},\ \frac{r^{\frac{1}{3}}}{x_3}$*. In particular, that implies Shevelev's property P.9.* □

Remark 4.2. Point 2. in the above theorem refines the statement of Theorem 4 from [10] by which there were at most 6 such families for each a.

Remark 4.3. By Property P.10 (page 185) we have $s = \frac{P(\gamma-1)P(2-\gamma)}{((\gamma-1)(2-\gamma))^2}$, which for $\mathbf{r = 1}$ is equivalent to the system

$$\begin{cases} P_s(x) := x^3 + (s-6)x^2 + 3x + 1 = 0 \\ x = (\gamma - 1)(\gamma - 2) \end{cases} \tag{28}$$

That gives some corollaries in view of the above results.

E.g. for $\mathbf{r = 1},\ \mathbf{s = 0}$ by Theorems 1.2 and 4.1 and Remark 1 we have

$$\pi_1(x; 1) = P(x) = x^3 - 3x + 1 = \prod_{k=1}^{3} \left(x - 2\cos\frac{2^k \pi}{9} \right)$$

and

$$\pi_2(x; 1) = x^3 - 3x^2 + 1 = \prod_{k=1}^{3} \left(x - \frac{1}{2\cos\frac{2^k\pi}{9}} \right).$$

Furthermore, for $\alpha = \frac{2k\pi}{9},\ k \in \mathbb{N},\ 3 \nmid k$ we obtain

$$\begin{aligned} P(1 - 2\cos\alpha) &= -1 + 4\cos^2\alpha(3 - 2\cos\alpha) = -1 + (2 + 2\cos 2\alpha)(3 - 2\cos\alpha) = \\ &= 6(1 + \cos 2\alpha - \cos\alpha) = 6\left(1 - 2\sin\frac{3}{2}\alpha\sin\frac{\alpha}{2}\right) = \\ &= 6\left(1 - (-1)^{\lfloor \frac{k}{3} \rfloor}\sqrt{3}\sin\frac{k\pi}{9}\right). \end{aligned}$$

and in the sequel we get

$$P\left(1 - 2\cos\frac{2^{k+1}\pi}{9}\right) = 6\left(1 + (-1)^k\sqrt{3}\sin\frac{2^k\pi}{9}\right), \quad k \in \mathbb{N},$$

whence, because of the form of roots of RCPs, we deduce

$$P_0(x) = x^3 - 6x^2 + 3x + 1 = \prod_{k=1}^{3} \left(x - 2 - (-1)^k 2\sqrt{3} \sin \frac{2^k \pi}{9} \right) =$$
$$= \left(x - 2 + 2\sqrt{3} \sin \frac{2\pi}{9} \right) \left(x - 2 - 2\sqrt{3} \cos \frac{\pi}{18} \right) \left(x - 2 + 2\sqrt{3} \sin \frac{\pi}{9} \right),$$

$$P_0(x+2) = x^3 - 9x + 9 = \prod_{k=1}^{3} \left(x + 2\sqrt{3}(-1)^{k+1} \sin \frac{2^k \pi}{9} \right) =$$
$$= \left(x + 2\sqrt{3} \sin \frac{2\pi}{9} \right) \left(x - 2\sqrt{3} \cos \frac{\pi}{18} \right) \left(x + 2\sqrt{3} \sin \frac{\pi}{9} \right).$$

Next, for $\mathbf{r} = \mathbf{1}$, $\mathbf{s} = \mathbf{1}$ we have

$$\pi_1(x;1) = x^3 - \frac{1}{\varphi^2} x^2 - \varphi^2 x + 1 = (x - \varphi)(x + \varphi) \left(x - \frac{1}{\varphi^2} \right),$$

and

$$\pi_2(x;1) := x^3 - \varphi^2 x^2 - \frac{1}{\varphi^2} x + 1 = \left(x - \frac{1}{\varphi} \right) \left(x + \frac{1}{\varphi} \right) (x - \varphi^2).$$

where φ is the golden ratio. We emphasize that these are the only known RCPs for which Shevelev's identities (see Property P.6, page 185) are trivial.

Moreover, from Cardano-type formulas in [14] we can obtain the following trigonometric form of the roots of $P_1(x) = x^3 + 5x^2 + 3x + 1$:

$$\frac{5}{3} + \frac{8}{3} \cos \left(\frac{1}{3} \arctan \frac{3\sqrt{15}}{11} \right) = 2 + \sqrt{5},$$
$$\frac{5}{3} - \frac{8}{3} \sin \left(\frac{\pi}{6} \pm \frac{1}{3} \arctan \frac{3\sqrt{15}}{11} \right) = \begin{cases} 2 - \sqrt{5}, \\ 1 \end{cases}$$

which implies the relations

$$\cos \left(\frac{1}{3} \arctan \frac{3\sqrt{15}}{11} \right) - \sin \left(\frac{\pi}{6} + \frac{1}{3} \arctan \frac{3\sqrt{15}}{11} \right) = \frac{1}{4},$$
$$\sin \left(\frac{\pi}{6} - \frac{1}{3} \arctan \frac{3\sqrt{15}}{11} \right) = \frac{1}{4},$$
$$\cos \left(\frac{1}{3} \arctan \frac{3\sqrt{15}}{11} \right) = \frac{1}{8} + \frac{3\sqrt{5}}{8} = \frac{\varphi^4 - 3}{4} = \frac{3\varphi - 1}{4},$$
$$\sin \left(\frac{1}{3} \arctan \frac{3\sqrt{15}}{11} \right) = \frac{\sqrt{15} - \sqrt{3}}{8} = \frac{\sqrt{3}}{4\varphi},$$

where φ is the golden ratio.

Remark 4.4. From Theorem 4.1 we also get the following formulas

$$\left(3+\sqrt{9-4s}+4\sqrt{9-s}\cos\left(\frac{1}{3}\arctan\sqrt{\frac{27}{9-4s}}\right)\right)\times$$
$$\times\left(3-\sqrt{9-4s}+4\sqrt{9-s}\sin\left(\frac{\pi}{6}-\frac{1}{3}\arctan\sqrt{\frac{27}{9-4s}}\right)\right)=36,$$

$$\left(3+\sqrt{9-4s}-4\sqrt{9-s}\sin\left(\frac{\pi}{6}-\frac{1}{3}\arctan\sqrt{\frac{27}{9-4s}}\right)\right)\times$$
$$\times\left(3-\sqrt{9-4s}+4\sqrt{9-s}\sin\left(\frac{\pi}{6}+\frac{1}{3}\arctan\sqrt{\frac{27}{9-4s}}\right)\right)=36,$$

$$\left(3+\sqrt{9-4s}-4\sqrt{9-s}\sin\left(\frac{\pi}{6}+\frac{1}{3}\arctan\sqrt{\frac{27}{9-4s}}\right)\right)\times$$
$$\times\left(3-\sqrt{9-4s}-4\sqrt{9-s}\cos\left(\frac{1}{3}\arctan\sqrt{\frac{27}{9-4s}}\right)\right)=36.$$

5. Asymptotic properties for RCP2

For any RCP2, which is given by $p(x) = x^3 + 3\sqrt[3]{kr}x^2 - 3\sqrt[3]{(k+1)r^2}x + r$ (see formula (16)), with roots ξ_1, ξ_2, ξ_3 the following identities hold [11]

$$\begin{aligned}&\sqrt[3]{\frac{\xi_1}{\xi_2}}+\sqrt[3]{\frac{\xi_2}{\xi_1}}+\sqrt[3]{\frac{\xi_2}{\xi_3}}+\sqrt[3]{\frac{\xi_3}{\xi_2}}+\sqrt[3]{\frac{\xi_1}{\xi_3}}+\sqrt[3]{\frac{\xi_3}{\xi_1}}=\\&\qquad=\sqrt[3]{9\left(\sqrt[3]{k}-\sqrt[3]{k+1}\right)}+\sqrt[3]{9\left(\sqrt[3]{k}+1\right)\left(1-\sqrt[3]{k+1}\right)},\end{aligned}\tag{29}$$

$$\begin{aligned}&\sqrt[3]{\frac{\xi_1}{r}}+\sqrt[3]{\frac{\xi_2}{r}}+\sqrt[3]{\frac{\xi_3}{r}}=\\&=\sqrt[3]{-3\left(2+\sqrt[3]{k}+\sqrt[3]{9\left(\sqrt[3]{k}-\sqrt[3]{k+1}\right)}+\sqrt[3]{9\left(\sqrt[3]{k}+1\right)\left(1-\sqrt[3]{k+1}\right)}\right)},\end{aligned}\tag{30}$$

$$
\begin{aligned}
&\sqrt[3]{\frac{\xi_1\xi_2}{r^2}}+\sqrt[3]{\frac{\xi_1\xi_3}{r^2}}+\sqrt[3]{\frac{\xi_2\xi_3}{r^2}}=\\
&\overset{k\neq 1}{=}\frac{\left(\sqrt[3]{\xi_1}+\sqrt[3]{\xi_2}+\sqrt[3]{\xi_3}\right)^3-\xi_1-\xi_2-\xi_3+3\sqrt[3]{\xi_1\xi_2\xi_3}}{3\sqrt[3]{r^2}\left(\sqrt[3]{\xi_1}+\sqrt[3]{\xi_2}+\sqrt[3]{\xi_3}\right)}=\\
&\overset{k\neq 1}{=}\frac{\sqrt[3]{9}+\sqrt[3]{3\left(\sqrt[3]{k}-\sqrt[3]{k+1}\right)}+\sqrt[3]{3\left(\sqrt[3]{k}+1\right)\left(1-\sqrt[3]{k+1}\right)}}{\sqrt[3]{2+\sqrt[3]{k}+\sqrt[3]{9\left(\sqrt[3]{k}-\sqrt[3]{k+1}\right)}+\sqrt[3]{9\left(\sqrt[3]{k}+1\right)\left(1-\sqrt[3]{k+1}\right)}}},
\end{aligned}
\tag{31}
$$

where in all cases real roots of third order are considered. Hence the limit follows

$$
\begin{aligned}
&\lim_{k\to 1}\frac{\sqrt[3]{9}+\sqrt[3]{3\left(\sqrt[3]{k}-\sqrt[3]{k+1}\right)}+\sqrt[3]{3\left(\sqrt[3]{k}+1\right)\left(1-\sqrt[3]{k+1}\right)}}{\sqrt[3]{2+\sqrt[3]{k}+\sqrt[3]{9\left(\sqrt[3]{k}-\sqrt[3]{k+1}\right)}+\sqrt[3]{9\left(\sqrt[3]{k}+1\right)\left(1-\sqrt[3]{k+1}\right)}}}=\\
&\quad=-\sqrt[3]{3\left(\sqrt[3]{2}+1\right)}
\end{aligned}
$$

since for roots x_1, x_2, x_3 of the polynomial

$$p(x)=x^3+3x^2-3\sqrt[3]{2}x+1$$

we get (see [11])

$$\sqrt[3]{x_1x_2}+\sqrt[3]{x_1x_3}+\sqrt[3]{x_2x_3}=-\sqrt[3]{3\left(\sqrt[3]{2}+1\right)}.$$

We also note that

$$\sqrt[3]{3}=\left(\sqrt[3]{2}+1\right)\sqrt[3]{\sqrt[3]{2}-1},$$

whence

$$\sqrt[3]{\frac{x_1}{x_2}}+\sqrt[3]{\frac{x_2}{x_1}}+\sqrt[3]{\frac{x_1}{x_3}}+\sqrt[3]{\frac{x_3}{x_1}}+\sqrt[3]{\frac{x_2}{x_3}}+\sqrt[3]{\frac{x_3}{x_2}}=-3.$$

Moreover from (31) for $k=0$ and from formula (4) in [10] we get

$$\frac{\sqrt[3]{9}-\sqrt[3]{3}}{\sqrt[3]{2-\sqrt[3]{9}}}=\sqrt[3]{3\left(1-\sqrt[3]{9}\right)},$$

i.e.

$$\sqrt[3]{3}-1=\sqrt[3]{\left(\sqrt[3]{9}-1\right)\left(\sqrt[3]{9}-2\right)},$$

since the polynomial $q(x)=x^3-3x+1$ is RCP.

Now, from (30) and (31) the following asymptotic expression can be deduced

$$\sqrt[3]{\frac{\xi_1}{r}} + \sqrt[3]{\frac{\xi_2}{r}} + \sqrt[3]{\frac{\xi_3}{r}} = \sqrt[3]{-3\sqrt[3]{k}\left(1 - \frac{\sqrt[3]{9}}{\sqrt[9]{k}} + \frac{2}{\sqrt[3]{k}} + o\left(\frac{1}{\sqrt[3]{k}}\right)\right)}, \tag{32}$$

so we get

$$\frac{\left(\sqrt[3]{\xi_1} + \sqrt[3]{\xi_2} + \sqrt[3]{\xi_3}\right)^3}{\xi_1 + \xi_2 + \xi_3} = 1 - \frac{\sqrt[3]{9}}{\sqrt[9]{k}} + \frac{2}{\sqrt[3]{k}} + o\left(\frac{1}{\sqrt[3]{k}}\right).$$

We note that if x_1, x_2, x_3 are roots of a complex polynomial $x^3 + px^2 + qx + r$ and $r \neq 0$ then

$$\frac{x_1}{x_2} + \frac{x_2}{x_1} + \frac{x_1}{x_3} + \frac{x_3}{x_1} + \frac{x_2}{x_3} + \frac{x_3}{x_2} = -3 - \frac{5pq}{r}.$$

Hence for RCP2 given by formula (16), that is $x^3 + 3\sqrt[3]{kr}x^2 - 3\sqrt[3]{(k+1)r^2}x + r$, we obtain

$$\begin{aligned}\frac{\xi_1}{\xi_2} + \frac{\xi_2}{\xi_1} + \frac{\xi_1}{\xi_3} + \frac{\xi_3}{\xi_1} + \frac{\xi_2}{\xi_3} + \frac{\xi_3}{\xi_2} &= -3 - 5\frac{3\sqrt[3]{kr}\left(-3\sqrt[3]{(k+1)r^2}\right)}{r} = \\ &= -3 + 45\sqrt[3]{k(k+1)} = -3 + 45\sqrt[3]{k^2}\left(1 + \frac{1}{3k}\right) + o\left(\frac{1}{\sqrt[3]{k}}\right),\end{aligned}$$

which by (32) implies

$$\frac{\left(\sqrt[3]{\xi_1} + \sqrt[3]{\xi_2} + \sqrt[3]{\xi_3}\right)^3}{\frac{\xi_1}{\xi_2} + \frac{\xi_2}{\xi_1} + \frac{\xi_1}{\xi_3} + \frac{\xi_3}{\xi_1} + \frac{\xi_2}{\xi_3} + \frac{\xi_3}{\xi_2}} = -\frac{1}{5} - \frac{3}{5\sqrt[3]{3}\sqrt[9]{k^4}} + o\left(\frac{1}{\sqrt[9]{k^4}}\right).$$

Acknowledgements. The authors gratefully acknowledges the many helpful suggestions and remarks of the Referee.

Bibliography

1. Barbero S., Cerruti U., Murru N., Abrate M.: *Identities involving zeros of Ramanujan and Shanks cubic polynomials.* J. Integer Seq. **16** (2013), article 13.8.1.
2. Berndt B.C.: *Ramanujan's Notebooks, Part IV.* Springer-Verlag, New York 1994.
3. Berndt B.C., Bhargava S.: *Ramanujan – for Lowbrows.* Amer. Math. Monthly **100**, no.7 (1993), 644–656.
4. Berndt B.C., Chan H.H., Zhang L.C.: *Radicals and units in Ramanujan's work.* Acta Arith. **87** (1998), 145–158.
5. Berndt B.C., Choi Y.S., Kang S.Y.: *The problems submitted by Ramanujan to the Journal of the Indian Mathematical Society.* In: Continued Fractions: From Analytic Number Theory to Constructive Approximation, a volume dedicated to L.J. Lange, Contemporary Math. **236** (1999), 15–56.
6. Ramanujan S.: *Notebooks (2 volumes).* Tata Institute of Fundamental Research, Bombay 1957.
7. Shevelev V.: *Three Ramanujan's formulas.* Kvant **6** (1988), 52–55 (in Russian). English translation: Kvant Selecta **14** (1999), 139–144.
8. Shevelev V.: *On Ramanujan cubic polynomials.* South East Asian J. Math. & Math. Sci. **8** (2009), 113–122.

9. Wituła R.: *Ramanujan type trigonometric formulas: the general form for the argument* $\frac{2\pi}{7}$. J. Integer Seq. **12** (2009), article 09.8.5.
10. Wituła R.: *Full description of Ramanujan cubic polynomials.* J. Integer Seq. **13** (2010), article 10.5.7.
11. Wituła R.: *Ramanujan cubic polynomials of the second kind.* J. Integer Seq. **13** (2010), article 10.7.5.
12. Wituła R.: *Ramanujan type trigonometric formulae.* Demonstratio Math. **45**, no. 4 (2012), 779–796.
13. Wituła R., Słota D.: *Cardano's formula, square roots, Chebyshev polynomials and radicals.* J. Math. Anal. Appl. **363** (2010), 639–647.
14. Wróbel A., Hetmaniok E., Pleszczyński M., Wituła R.: *On an improvement of the numerical application for Cardano's formula in Mathematica software.* In: Proceedings of the International Symposium for Young Scientists in Technology, Engineering and Mathematics, vol. 1543, 71–78, 2015. On-line access via http://ceur-ws.org/Vol-1543/.